# ON THE DISTRIBUTION OF THE STARS IN SPACE ESPECIALLY IN THE HIGH GALACTIC LATITUDES[1]

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ABSTRACT

*Distribution of stars in space.*—While the results reported in this paper are provisional, in that some recently available parallaxes are not included and the investigation of the lower galactic latitudes has not been completed, it is unlikely that more complete data will materially modify them. (1) *In galactic latitudes* $\pm 40°$ *to* $\pm 90°$. The *average parallax for all stars of a given magnitude* ($m$) *and proper motion* ($\mu$) is found to be represented satisfactorily by the formula $\log \pi = -0.691 - 0.0682\,m + 0.645 \log \mu$. By combining this result with the law of dispersion of parallax for stars of given $m$ and $\mu$, which had already been determined, the two fundamental laws which determine the arrangement of stars in space were found. The first of these is the *luminosity-curve* or frequency of the several absolute magnitudes per unit volume, which, at least near the sun, is found to correspond accurately, from $-10^{M}_{.}6$ to $+7^{M}_{.}4$, to a symmetrical probability-curve with the equation: $\log \phi(M) = -2.394 + 0.1858M - 0.03450M^2$ (see Table IV and Fig. 1). Assuming that the luminosity-curve is the same for all distances from the sun, the median absolute magnitude of all stars is 2.7 (which is about 2.9 magnitudes fainter than the sun), with a probable error of $\pm 1^{M}_{.}69$. The second law is the *law of stellar densities* as a function of parallax. For distances below 1000 parsecs the densities can be determined directly (see Table V). For greater distances the following formula was derived from the luminosity-curve and the distribution-curve of apparent magnitudes, assuming no extinction of light in space: $\log \Delta(\rho) = -3.425 + 3.526 \log \rho - 0.943 (\log \rho)^2$ (see Table VI). (2) *In galactic latitudes* 0°, 30°, 60°, *and* 90°. A less refined investigation of all stars in these latitudes has led to formulae for the densities as a function of parallax which are similar to that given above (see Tables VI and VII). These results enable us to draw a *section of the galactic system* at right angles to the galactic plane, with the sun at the center (Fig. 2), which shows that in the direction of the galactic poles about 1500 parsecs may be taken as practically the limit of the system, while in a direction in the plane of the Milky Way the same small density is eight times more distant. The authors point out that, since symmetry around the galactic poles is assumed, this work is merely a second approximation to the complete solution of the problem.

1. The investigations contained in *G.P.*[3] 27, 29, and 30 were carried out for the purpose of making possible an elaborate treatment of the arrangement of the stars in space. At least two more

[1] *Contributions from the Mount Wilson Observatory*, No. 188.

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[3] By *G.P.* we will denote the *Publications of the Astronomical Laboratory at Groningen*.

publications will be necessary to complete this investigation. Now that, after so many years of preparation, our data seem at last to be sufficient for the purpose, we have been unable to restrain our curiosity and have resolved to carry through completely a small part of the work, even though, by so doing, the rules for strict economy of labor cannot be altogether adhered to.

The present paper is the outcome of this more or less provisional work. In the main it relates to the stars as a whole between galactic latitudes $\pm 40^\circ$ and $\pm 90^\circ$. It is provisional in that the extremely valuable parallaxes now placed at our disposal by Mitchell have not yet been used, and moreover because the thorough discussion which will have to be made of the measured parallaxes in all galactic latitudes is not yet completed. We expect little to be changed, however, in the definitive treatment of the subject, which will form a part of the more comprehensive work.

A somewhat less refined investigation of all the stars (irrespective of spectrum) in galactic latitudes 0°, 30°, 60°, and 90° has also been added.

2. In *G.P.* 30 were found the numbers of stars for each magnitude down to $m=12.0$ and for every value of the proper motion $\mu$. These numbers, corrected for observational errors, are contained in Table 19, which also gives for magnitudes 13 and 14 the numbers for proper motions exceeding 0".200.

We begin by deriving the average parallax for each of these classes of stars, i.e., for stars having given values of $m$ and $\mu$. Following the method indicated in *G.P.* 8, we tentatively start from the formula arrived at in that publication,

$$\log \pi_{m,\mu} = A + Bm + C \log \mu . \qquad (1)$$

It turns out that this formula represents all our data satisfactorily. It is true that some of the divergences are larger than seems desirable, but there is reason to believe that the defect is not in the formula. At all events, the question whether a still more satisfactory formula can be assigned can be settled only by the later definitive treatment of the data for the whole of the sky.

The measured parallaxes available for our investigation embrace all the published results that seem to deserve confidence, excluding those whose probable errors exceed 0".025. In addition we are indebted to Professor Schlesinger for a magnificent list of unpublished determinations. We cordially thank him for this generous act of courtesy. Professor Mitchell, too, as already mentioned, has communicated his invaluable results. These will, of course, play a prominent part in our more elaborate work, but they arrived too late for the present note.

For those cases in which several determinations are available for the same star, weights were adopted, not exactly in accordance with the probable errors as given by the authors, but modified somewhat by the supposition that small systematic errors still attach to all the results.

For proper motions exceeding 1".00, the stars in all galactic latitudes were used. For smaller motions we confined ourselves for the present to galactic latitudes higher than 40°.

In order to find the change of $\pi$ with $\mu$ (the constant $C$ in (1)) we divided the stars with measured parallaxes into two groups according to brightness—magnitudes 3.5 to 6.5, and those that are fainter. Both these groups were subdivided according to proper motion. We thus obtained the normal values in Tables I and II, which, by means of small corrections derived from *G.P.* 8, were all reduced to magnitudes 5.0 and 8.0.

The data obtained from direct parallax determinations given in these tables were supplemented by results derived from the parallactic motions in Table 25 of *G.P.* 29. The latter parallaxes were obtained by adopting 19.5 km for the sun's velocity, and are entered in the last lines of Tables I and II.

The constants in formula (1) were then determined in such a way that the values from the parallactic motions are almost exactly represented, while at the same time the measured parallaxes are represented as well as possible. The results thus found are

$$\log \pi_{5.0} = -1.040 + 0.630 \log \mu \tag{2}$$

$$\log \pi_{8.0} = -1.163 + 0.659 \log \mu . \tag{3}$$

The parallaxes computed with these formulae have been designated by $\pi_1$ in Tables I and II. Although the residuals $O - \pi_1$

may not seem to be all that can be desired, we believe our tables prove that, at least as far as the change with proper motion is concerned, formula (1) cannot be materially in error. As we have already remarked, however, it seems best to reserve a closer discussion of this point until later. As the combined result of (2) and (3) we adopt

TABLE I

MAGNITUDE 3.5 TO 6.5

| $m$ | True $\mu$ | $\bar{\pi}$ | No. | $\pi_1$ | $\pi_2$ | $O-\pi_1$ | $O-\pi_2$ |
|---|---|---|---|---|---|---|---|
| 5.0..... | 0".048 | +0".018 | 31 | 0".0135 | 0".013 | +0".0045 | +0".005 |
| 5.0..... | .123 | + .034 | 23 | .0245 | .024 | + .0095 | + .010 |
| 5.0..... | .251 | + .043 | 27 | .038 | .038 | + .005 | + .005 |
| 5.0..... | .436 | + .047 | 15 | .054 | .054 | − .007 | − .007 |
| 5.0..... | .670 | + .072 | 18 | .071 | .072 | + .001 | .000 |
| 5.0..... | 1.260 | + .134 | 23 | .105 | .108 | + .029 | + .026 |
| 5.0..... | 3.77 | + .174 | 9 | .211 | .218 | − .037 | − .044 |
| 5.0 (mean) *G.P.* 29.. | | +0.0204 | ......... | 0.0200 | 0.0192 | +0.0004 | +0.0012 |

TABLE II

MAGNITUDE 6.5 AND FAINTER

| $m$ | True $\mu$ | $\bar{\pi}$ | No. | $\pi_1$ | $\pi_2$ | $O-\pi_1$ | $O-\pi_2$ |
|---|---|---|---|---|---|---|---|
| 8.0..... | 0".202 | +0".021 | 29 | +0".024 | 0".021 | −0".003 | 0".000 |
| 8.0..... | .486 | + .032 | 22 | + .043 | .036 | − .011 | − .004 |
| 8.0..... | .714 | + .040 | 31 | + .055 | .047 | − .015 | − .007 |
| 8.0..... | 1.30 | + .085 | 28 | + .082 | .069 | + .003 | + .016 |
| 8.0..... | 4.05 | + .214 | 13 | + .173 | .143 | + .041 | + .071 |
| 8.0 (mean) *G.P.* 29.. | | +0.0079 | ......... | 0.0079 | 0.0072 | 0.0000 | +0.0007 |

$$C=+0.645. \tag{4}$$

3. We derive the constants $A$ and $B$ exclusively from the mean parallaxes found from parallactic motions. Writing (1), in which $C$ has the value (4), in the form

$$\pi=10^{A+Bm}\times 10^{0.645 \log \mu}, \tag{5}$$

we find for the mean parallax of all the stars of magnitude $m$,

$$\bar{\pi} = 10^{A+Bm} \times 10^{\overline{0.645 \log \mu}}, \qquad (6)$$

where the dashes indicate average values.

Since the total number of stars for every value of $m$ and $\mu$ is given in Table 19 of *G.P.* 30, we can at once find the average value occurring in the second member. The values of $\bar{\pi}$ are obtained from Table 25 (in which the sun's velocity = 19.5 km). Introducing logarithms, we find the equations of condition in Table III.

TABLE III

EQUATIONS OF CONDITION

| Magnitude | Equation | $\log \pi_{\text{comp.}}$ | $O-C$ $\log \pi_{29} - \log \pi_{\text{comp.}}$ |
|---|---|---|---|
| 4 . . . . . . . . . . . . . . . . | $A+4B=-1.033$ | $-0.975$ | $-0.058$ |
| 5 . . . . . . . . . . . . . . . . | $A+5B=-1.019$ | $-1.046$ | $+.027$ |
| 6 . . . . . . . . . . . . . . . . | $A+6B=-1.121$ | $-1.118$ | $-.003$ |
| 7 . . . . . . . . . . . . . . . . | $A+7B=-1.156$ | $-1.189$ | $+.033$ |
| 8 . . . . . . . . . . . . . . . . | $A+8B=-1.215$ | $-1.260$ | $+.045$ |
| 9 . . . . . . . . . . . . . . . . | $A+9B=-1.326$ | $-1.332$ | $+.006$ |
| 10 . . . . . . . . . . . . . . . . | $A+10B=-1.429$ | $-1.403$ | $-.026$ |
| 11 . . . . . . . . . . . . . . . . | $A+11B=-1.499$ | $-1.474$ | $-0.025$ |

Solved by least squares, these equations yield

$$A = -0.690 \qquad B = -0.0713, \qquad (7)$$

so that finally

$$\log \pi_{m,\mu} = -0.690 - 0.0713\, m + 0.645 \log \mu, \qquad (8)$$

with which the third column in Table III was computed. Instead of this formula, however, the one actually used in what follows is

$$\log \pi_{m,\mu} = -0.691 - 0.0682\, m + 0.645 \log \mu. \qquad (9)$$

Since the two equations represent the observations almost equally well, we have not repeated the computations.

The coefficient of $m$ corresponds to log $\epsilon$ in *G.P.* 8, formula (3). Hence from (9), $\epsilon = 0.855$, whereas in *G.P.* 8 the value $\epsilon = 0.905$ was adopted. In the meantime this value of $\epsilon$ has already been corrected in *G.P.* 11 (p. 20) to $\epsilon = 0.87$, in good agreement with the present value.

The parallaxes obtained by means of (9) are given in Tables I and II under the heading $\pi_2$. The residuals for this definitive solution are given under the heading $O - \pi_2$. Only in the case of the largest proper motions are they somewhat excessive. That this is not due to a defect in our formula is proved by the fact that the sign in the case of magnitude 8.0 is the opposite of that for magnitude 5.0. For the present purpose we do not consider these residuals for the largest proper motions to be of prime importance, since they are extremely rare.[1] Had this not been the case we might have lessened these extreme values somewhat by adopting a slightly larger value of $C$.

4. Formula (9) contains the solution of the problem presented in section 2, viz., to find the average parallax for each of the classes of stars of given $m$ and $\mu$ contained in Table 19 of *G.P.* 30. The stars of apparent magnitudes 13 and 14 in this table, all having large proper motions, must be of extremely low absolute magnitude; they are accordingly very valuable for extending the luminosity-curve at its fainter extremity. That the frequency of the smaller proper motions is not known for these stars is of small importance. Their influence on the computations actually used for what follows is so small that we can quite safely introduce extrapolated values.

Although by (9) we can find the *average* parallax of the stars of any $m$ and $\mu$, it is of course evident that the parallax of any single star of the given $m$ and $\mu$ will in general differ from the mean parallax. But for the problem of the general arrangement of the stars in space it is not necessary to know the distance of each individual star. It suffices to know how many stars have such and such a distance. This knowledge may be obtained if, in addition to the average parallaxes already found, we can find the dispersion law of the parallaxes, that is, the law which, for the stars of given

[1] In further computations the best plan will probably be to use for these stars the individual parallaxes as found by direct determination.

$m$ and $\mu$, specifies, for any value of $a$, the frequency of a parallax $a$ times the average parallax.

Having found the number of stars of given magnitude $m$ and a given parallax, we shall know of course for these same stars the absolute magnitude $M$.

It has been shown in *G.P.* 11 (pp. 17–20) that widely differing assumptions as to the dispersion law lead to results that differ but little. In this preliminary solution, therefore, we have not deemed it necessary to derive this law anew, but have adopted the one found and tabulated in *G.P.* 8 (solution D).

Since it thus becomes possible to find, for any given distance, the number of stars of each absolute magnitude, we can evidently derive the two fundamental laws which determine the arrangement of the stars in space, viz., the luminosity-curve and the law of stellar densities. A convenient manner of conducting the computations is explained at length in *G.P.* 11 and need not be repeated here. It should be remembered, however, that it introduces the two assumptions: (1) there is no appreciable extinction of light in space; (2) the frequency of the several absolute magnitudes (luminosity-curve) does not change with the distance from the sun. These assumptions can be avoided, at least to a considerable extent. They will be discussed in the definitive solution.

Different units of distance have been used by different astronomers. That most widely used at present is the parsec. For the sake of uniformity we have resolved henceforth not only to use this unit but also to use the name, which is very convenient (though very ugly). To conform with this new unit, the value of the absolute magnitude, which must still be defined to be the magnitude of a star as it would appear at the unit of distance, must also be changed. Its numerical value will now be five less than it was according to former publications, and may be found by the formula

$$M = m + 5 \log \pi = m - 5 \log \rho. \quad \text{(A)}$$

The results obtained in the present case are in Tables IV and V. The table giving the densities will presently be extended to greater distances through the consideration of other data.

Table V will presently be extended to greater distances by the consideration of other data.

TABLE IV

LOG NUMBER OF STARS PER CUBIC PARSEC NEAR THE SUN

(Determining Luminosity-Curve)

| $M$ | log $\phi(M)$ | Computed | O−C |
|---|---|---|---|
| −11.64 | 1.2 −10 | 0.8 −10 | +0.4 |
| −10.64 | 1.6 | 1.7 | − .1 |
| − 9.64 | 2.55 | 2.61 | − .06 |
| − 8.64 | 3.373 | 3.425 | − .052 |
| − 7.64 | 4.148 | 4.173 | − .025 |
| − 6.64 | 4.843 | 4.851 | − .008 |
| − 5.64 | 5.474 | 5.461 | + .013 |
| − 4.64 | 6.020 | 6.001 | + .019 |
| − 3.64 | 6.493 | 6.473 | + .020 |
| − 2.64 | 6.894 | 6.875 | + .019 |
| − 1.64 | 7.215 | 7.209 | + .006 |
| − 0.64 | 7.472 | 7.473 | − .001 |
| + 0.36 | 7.662 | 7.669 | − .007 |
| + 1.36 | 7.776 | 7.795 | − .019 |
| + 2.36 | 7.836 | 7.853 | − .017 |
| + 3.36 | 7.819 | 7.841 | − .022 |
| + 4.36 | 7.737 | 7.761 | − .024 |
| + 5.36 | 7.627 | 7.611 | + .016 |
| + 6.36 | 7.364 | 7.393 | − .029 |
| + 7.36 | 7.200 | 7.105 | + .095 |
| + 8.36 | 7.00 | 6.75 | + .25 |
| + 9.36 | 6.4 | 6.3 | +0.1 |

TABLE V

DENSITY (=1 NEAR SUN)

| $\pi$ | Density | $\pi$ | Density |
|---|---|---|---|
| 0\".00118 | 0.089 | 0\".0296 | 0.918 |
| .00187 | .179 | .0469 | 1.000 |
| .00296 | .298 | .0743 | 1.000 |
| .00469 | .451 | .118 | 1.000 |
| .00743 | .600 | .187 | 1.000 |
| .0118 | .760 | 0.296 | 1.000 |
| 0.0187 | 0.864 | ............ | ............ |

The result for the luminosity-curve may be considered to be the final result of this paper. Its general high accuracy appears in the agreement of the large overlapping portions of the independent curves found for different distances from the sun. For brevity's sake we omit the numbers which show this, but even without them we gain an insight into the accuracy through the astonishingly close approach of the values in Table IV to the simple analytical curve

$$\log \phi(M) = -2.394 + 0.1858\, M - 0.03450\, M^2 . \qquad (10)$$

The last column of the table shows the residuals.

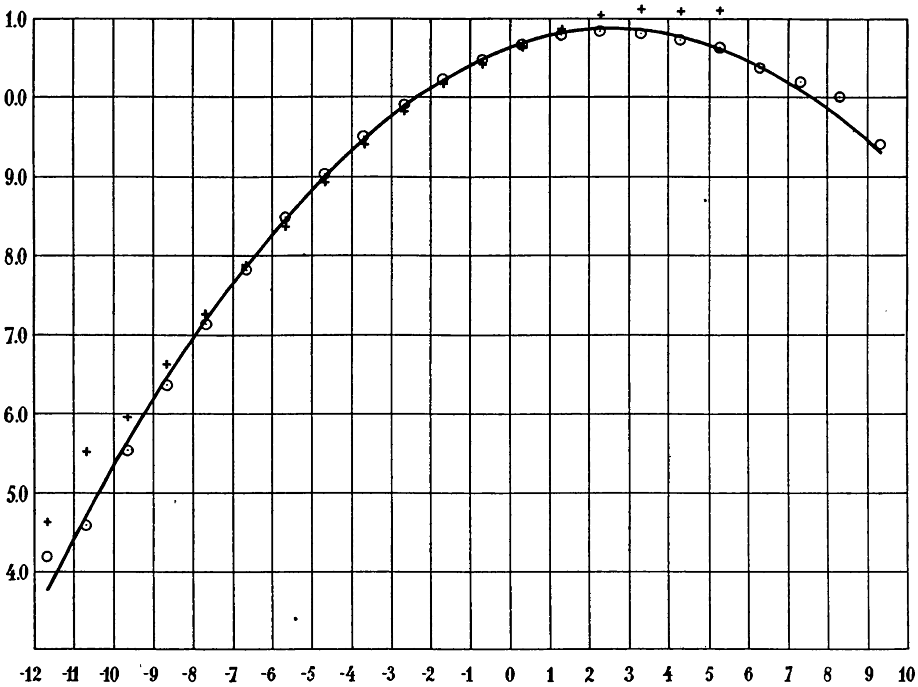


FIG. 1.—Luminosity-curve, all spectral types together. Abscissae are absolute magnitudes (unit of distance, 1 parsec). Ordinates are logarithms of numbers of stars per 1000 cubic parsecs near the sun.

A still better appreciation of the close approach of theory and observation may be obtained by inspecting Figure 1, which shows the curve (10) and the observed values, represented by small circles. With absolute magnitudes as abscissae, this curve gives the logarithm of the number of stars per 1000 cubic parsecs, instead of per cubic parsec, as does formula (10).

We may write (10) in the form

$$\phi(M) = A \frac{h}{\sqrt{\pi}} e^{-h^2(M-M_0)^2} \qquad (11)$$

where

$$\left.\begin{array}{c} A=0.0451,\ M_0=2.693,\ h=0.2818 \\ r=\dfrac{0.4769}{h}=1.692 \end{array}\right\} . \qquad (12)$$

It thus appears that the luminosity-curve (or rather the absolute-magnitude curve) is represented very closely by an error-curve spread round the median value $M=2.7$, with a probable error of $\pm 1.69$. If we assume that the agreement of the two curves holds for absolute magnitudes beyond those afforded by our data, the total number of stars per cubic parsec in the vicinity of the sun from the very brightest, absolutely, to the faintest is 0.0451.

The crosses in Figure 1 represent the results found in *G.P.* 11 from wholly different and very scanty material.[1] In *G.P.* 11 only the data for stars brighter than 6.0 or 6.5 may be considered as fairly reliable, whereas in the present paper most of the data are certainly trustworthy down to magnitude 11 (inclusive), and even down to magnitude 12 (inclusive) for the all-important proper motions of considerable amount, and begin to fail altogether only beyond magnitude 14.

Although the present curve is therefore infinitely preferable to the old one, it is still eminently satisfactory to see how closely the old curve agrees with the present result. The extremities were of necessity unreliable; the data on which it was based were defective not only in the proper motions and magnitudes but even more so in the measured parallaxes.

The most gratifying advance is the determination of the maximum of the curve. Owing to the inclusion of material for stars as faint as twelfth, thirteenth, and fourteenth magnitudes, we are at last well beyond the maximum, which can now be located with considerable precision. In many respects this will prove a very important gain, indeed, since it allows us for the first time to express with some certainty the relation between the average absolute magnitude of the stars and that of the sun. The absolute magni-

[1] Reduced from the Potsdam scale to the Harvard scale by subtracting 0.16 from the magnitudes.

tude of the sun[1] appears to be near −0.2, which is 2.9 magnitudes *brighter* than the average magnitude of all the stars.

If in Table IV we disregard the first two and the last two values, which are of necessity very uncertain, there still remain values ranging over no less than eighteen magnitudes, that is, over luminosities varying from one to sixteen millions, all of which lie with extreme precision on the same Gaussian curve. We doubt whether any other case is known in the whole domain of science of so close an agreement over so large a range—a range of more than ten times the probable error. It is difficult to avoid the conclusion that we have here to do with a law of nature, a law which plays a dominant part in the most diverse natural phenomena. This in itself would lead us to assume that the curve (11) will still hold for the remainder of the descending branch of the luminosity-curve. There is at least one other fact which tends to confirm our belief that this will be the case. For the B0, B1, B2, and B3 stars practically the whole curve is covered by the observations,[2] and no assured deviation from an error curve has been found. It is true that in this case the number of observations is very small. Be this as it may, it would be more satisfactory if the curve could still be prolonged somewhat by observation. We hope to consider elsewhere the possibility of doing so.

5. The densities in Table V do not extend to distances greater than a thousand parsecs and cannot claim any very great accuracy even for distances somewhat below this limit. It is evident that much would be gained by introducing a consideration of the known total number of stars of any given magnitude $m$.[3]

As before let $\phi(M)dM$ represent the luminosity-curve, i.e., the number of stars per cubic parsec near the sun having absolute magnitudes between $M$ and $M+dM$. Further let $N_m dM$ represent the number of stars per 10,000 square degrees having apparent magnitudes $m$ to $m+dm$. Finally let $\Delta(\rho)$ represent the total number of stars of any absolute magnitude per cubic parsec, taking

[1] *Astrophysical Journal*, **43**, 105, 1916.

[2] *Mt. Wilson Contr.*, No. 147, p. 72; for the B3 stars in particular see Table XXXIII.

[3] See *Proceedings Roy. Acad. Amsterdam*, March, 1908.

as unit the number in the neighborhood of the sun. It is easy to show, $\rho$ being the distance, that

$$N_m = 0.9696\pi \int_0^\infty \rho^2 \Delta(\rho)\phi(m-5\log\rho)d\rho \ . \tag{13}$$

Schwarzschild[1] has shown that if

$$\phi(M) = e^{p+qM+rM^2} \tag{14}$$

and

$$N_m = e^{a+bm+cm^2} \tag{15}$$

we shall have

$$\Delta(\rho) = e^{h+k\log\rho+l(\log\rho)^2} \ . \tag{16}$$

Since (10) or (11) shows that $\phi(M)$ has indeed the form (14), and since it was long ago shown (see for example *G.P.* 27, p. 24) that with astonishing approximation $N_m$ has also the form (15), we can find the density $\Delta(\rho)$ at once from (16). The relation between the constants is found to be

$$l = \frac{25cr}{r-c} , \tag{17}$$

and further, if we write

$$G = -l - 25r \tag{18}$$

$$k = 5q - 6.9078 - \frac{G(b-q)}{5r} \tag{19}$$

$$h = a - p - 2.5203 + \frac{\log G}{0.8686} - \frac{(k-5q+6.9078)^2}{4G} . \tag{20}$$

6. By means of these formulae we have computed the densities from the values of $N_m$ in Table V of *G.P.* 27. Since the labor involved is small, we were not able to resist the temptation to treat in the same way the data for galactic latitudes 0°, 30°, 60°, and 90°, for this must give a first insight into the arrangement of the

[1] *Astronomische Nachrichten*, 185, 81, 1910.

stars of the whole stellar system in space. In executing this plan we had of course to assume that the luminosity-curve is the same for all galactic latitudes, which seems little doubtful, although naturally we intend to investigate the matter in our definitive solution.

A little preliminary computation gave the following values of the constants by which the values of $N_m$ in *G.P.* 27 are excellently represented:

| Galactic Latitude | 0° | 30° | 60° | 90° | 40° to 90° | Luminosity-Curve |
|---|---|---|---|---|---|---|
| *a*....... | −1.249 | −1.296 | −1.748 | −2.079 | −1.605 | $p = -5.512$ |
| *b*....... | +1.668 | +1.593 | +1.692 | +1.783 | +1.659 | $q = +0.4279$ |
| *c*....... | −0.0325 | −0.0343 | −0.0438 | −0.0509 | −0.0415 | $r = -0.07944$ |

The constants of the luminosity-curve in accordance with (10) have also been added. Using these data we find by means of formulae (17) to (20):

| Galactic Latitude | 0° | 30° | 60° | 90° | 40° to 90° |
|---|---|---|---|---|---|
| *h*........... | −5.830 | −5.193 | −9.221 | −14.319 | −7.885 |
| *k*........... | +5.705 | +5.481 | +9.320 | +14.092 | +8.120 |
| *l*........... | −1.366 | −1.508 | −2.441 | − 3.542 | −2.171 |

which lead to the very convenient formulae

$$\left.\begin{array}{lll} \text{Gal. Lat. } 0°, & \log \Delta(\rho) = -2.532 + 2.478 \log \rho - 0.593 (\log \rho)^2 \\ 30, & \phantom{\log \Delta(\rho)} = -2.256 + 2.381 \phantom{\log \rho} - 0.655 \\ 60, & \phantom{\log \Delta(\rho)} = -4.005 + 4.048 \phantom{\log \rho} - 1.060 \\ 90, & \phantom{\log \Delta(\rho)} = -6.219 + 6.120 \phantom{\log \rho} - 1.538 \\ 40 \text{ to } 90, & \phantom{\log \Delta(\rho)} = -3.425 + 3.526 \phantom{\log \rho} - 0.943 \end{array}\right\} \quad (21)$$

The evident and well-recognized defect of these solutions is that the formula gives $\Delta(0) = 0$, which is inadmissible. The values of $\Delta(\rho)$ furnished by (16) or (21) for the smaller distances cannot therefore be accepted. Fortunately our former solution gives good values for these very distances. The comparison of the results obtained by the two methods, which can be made for galactic latitudes 40° to 90° and which is given below, proves that the

TABLE VI

LOG $\Delta(\rho)$

| LOG $\rho$ | $\rho$ IN PARSECS | LATITUDE | | | | LATITUDE 40° TO 90° | | |
|---|---|---|---|---|---|---|---|---|
| | | 0° | 30° | 60° | 90° | Formula (16) | Table V | Adopted |
| 1.0...... | 10.0 | 0.00 | 0.00 | 0.00 | 0.00 | 9.16 | 0.00 | 0.00 |
| 1.2...... | 15.8 | .00 | .00 | .00 | .00 | .45 | .00 | .00 |
| 1.4...... | 25.1 | .00 | .00 | .00 | .00 | .66 | 9.99 | 9.99 |
| 1.6...... | 39.8 | .00 | .00 | .00 | .00 | .80 | .96 | .96 |
| 1.8...... | 63.1 | .00 | .00 | .00 | .00 | .87 | .92 | .92 |
| 2.0...... | 100 | .00 | 9.89 | 9.85 | 9.87 | .85 | .84 | .84 |
| 2.2...... | 158 | .00 | .81 | .77 | .80 | .77 | .73 | .75 |
| 2.4...... | 251 | .00 | .68 | .60 | .61 | .61 | .59 | .60 |
| 2.6...... | 398 | 9.90 | .51 | .35 | .30 | .37 | .39 | .38 |
| 2.8...... | 631 | .76 | .28 | .02 | 8.86 | .06 | .14 | .08 |
| 3.0...... | 1000 | .56 | 8.99 | 8.60 | .30 | 8.67 | ........ | 8.67 |
| 3.2...... | 1580 | .32 | .66 | 8.10 | 7.62 | 8.20 | ........ | 8.20 |
| 3.4...... | 2510 | .04 | .27 | 7.50 | 6.81 | 7.66 | ........ | 7.66 |
| 3.6...... | 3980 | 8.70 | 7.83 | 6.83 | 5.88 | 7.05 | ........ | 7.05 |
| 3.8...... | 6310 | .32 | .33 | 6.07 | 4.83 | 6.36 | ........ | 6.36 |
| 4.0...... | 10000 | 7.89 | 6.79 | 5.23 | 3.65 | 5.59 | ........ | 5.59 |
| 4.2...... | 15800 | 7.42 | 6.19 | 4.30 | 2.35 | 4.75 | ........ | 4.75 |
| 4.4...... | 25100 | 6.89 | 5.54 | 3.28 | 0.93 | 3.83 | ........ | 3.83 |
| 4.6...... | 39800 | 6.32 | 4.84 | 2.19 | 9.39 | 2.84 | ........ | 2.84 |
| 4.8...... | 63100 | 5.70 | 4.08 | 1.00 | 7.72 | 1.77 | ........ | 1.77 |
| 5.0...... | 100000 | 5.03 | 3.27 | 9.74 | 5.93 | 0.63 | ........ | 0.63 |

TABLE VII

$\rho$

| $\Delta(\rho)$ | 0° | 30° | 60° | 90° |
|---|---|---|---|---|
| 1.00............ | 0 | 0 | 0 | 0 |
| 0.40............ | 910 | 320 | 250 | 250 |
| .16............ | 1450 | 710 | 490 | 450 |
| .063............ | 3550 | 1320 | 800 | 660 |
| .025............ | 5750 | 2140 | 1200 | 910 |
| .010............ | 8910 | 3310 | 1700 | 1230 |
| .0040........... | 13200 | 4900 | 2340 | 1580 |
| .0016........... | 19100 | 7080 | 3090 | 2000 |
| .00063.......... | 26900 | 10000 | 4070 | 2510 |
| .00025.......... | 37200 | 13500 | 5130 | 3090 |
| 0.00010.......... | 50100 | 18200 | 6460 | 3720 |

present method yields good results as soon as the maximum is well passed.

For distances less than 100 parsecs[1] we have accordingly adopted the first solution for galactic latitudes 40° to 90.° For distances beyond this limit the two solutions give results which agree surprisingly well, and we have adopted the mean of the two. Beyond 630 parsecs the second solution is the only one available. For galactic latitudes 0°, 30°, 60°, and 90° the first solution has not been carried out. In close agreement with the result just found for the high galactic latitudes, we have assumed the density to be constant up to the maximum of the second solution, while, from that value on, the second solution was adopted. Table VI shows the values thus obtained, from which by interpolation we derive Table VII.

Figure 2 was constructed with the aid of Table VII. If we imagine this figure completed by the addition of its reflected image on the other side of the line *AB*, it will represent a complete section through the galactic system at right angles to the galactic plane. The lines shown are lines of constant density. The sun lies at the center *S*. The numbers along the bottom line show the distances expressed in parsecs. The density is

[1] By the substitution, below the maximum, of the first solution for the second, the total number of stars $N_m$ which the theory yields exceeds somewhat the total found from observation. But since the volume of space below the maximum of formula (16) is very small, we have assumed that no serious error is introduced.

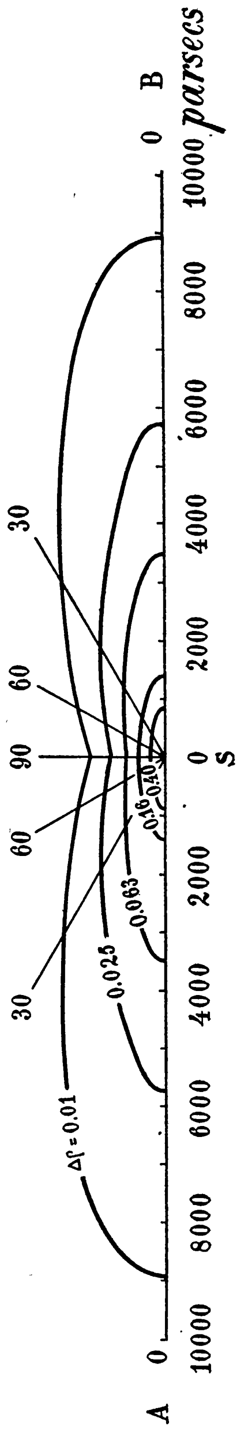


Fig. 2.—Distribution of density in a plane perpendicular to the galactic circle passing through the center of the stellar system. AB is the plane of the galaxy; the marginal numbers 0, 30, 60, and 90 are galactic latitudes. The curves are lines of equal density, the density at the sun (assumed to be at the center of the system) being taken as unity.

unity at $S$, and 0.40, 0.16, 0.063, 0.025, and 0.01, respectively, along the lines shown in the figure.

Whether the inflection near the pole in the lines of small density is real we are not at the moment prepared to say. If it is not, the layers of small density become singularly flat for an enormous range in distance.

We have not yet studied the question of the limits to which our results for the density are fairly reliable; but there is hardly a doubt that we can safely adopt them as a good approximation up to at least 1500 parsecs. In the direction of the pole of the Galaxy this brings us to what many will be inclined to take as practically the limit of the system. At least the density at that distance cannot be 1/200 of that near the sun. In any direction along the plane of the Milky Way, on the contrary, this same limit must be eight times more distant.

Leaving further considerations for the definitive solution, we conclude by expressing the hope that our present attempt to derive the arrangement of the stars will not be misunderstood. We have always considered this problem to be one that must be solved by successive approximations. The first was that attempted in *G.P.* 11, in which the system was considered as a whole, without regard to the differences shown in different galactic latitudes. The present solution constitutes a second approximation. It assumes that the center lies near or in the sun; that the system is symmetrical with respect to the galactic plane, which is supposed to pass through the sun; and that the normal to the plane through the center is an axis of symmetry. In subsequent approximations these assumptions will be dropped. Although perhaps something in the right direction has already been accomplished, much still remains to be done, especially in the study of the galactic clouds, before the solution will be complete.

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